Unit-4/B

Time Varying Fields

Objectives:- Faraday's laws, electromotive force, $e = \oint \vec{E} \cdot d\vec{l}$, integral and differential form of Faraday's laws, self inductance of a current loop and solenoid, mutual inductance between two arbitrary circuits, relation between self and mutual inductances, Coefficient of coupling (9)

Maxwell's equations and their interpretation, energy in electrostatics and magnetic fields, energy in e.m fields, Poynting vector, linear momentum and its conservation in e.m fields. (9)

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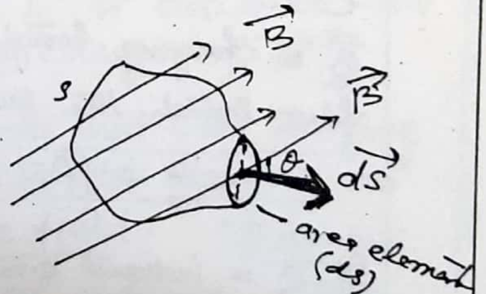
(1). **Electromagnetic Induction**:- "The phenomenon of production of electric current or e.m.f in a closed circuit (or coil) due to change of magnetic flux linked with it is called electromagnetic induction" The e.m.f so produced is called induced e.m.f and the current so produced is called induced current. The induced e.m.f and current in the coil last as long as the magnetic flux linked with it keeps on changing. This effect is just converse to magnetic effect of electric current and it was discovered by Michel Faraday in 1831.

Magnetic flux ϕ_m :- "The total number of magnetic lines of force passing normally through any surface is called magnetic flux."

Mathematically, if magnetic field $\vec{B}$ makes an angle θ with $d\vec{s}$ as shown in fig (1), then magnetic flux through this element is



$$d\phi_m = \vec{B} \cdot d\vec{s}$$

$\therefore$ Magnetic flux through the entire surface S is

$$\phi_m = \iint \vec{B} \cdot d\vec{s}.$$

S.I unit of magnetic flux is Weber (wb) or Tm^2

Its c.g.s unit is maxwell. And $1 wb = 10^8$ maxwell.

Magnetic flux density (or magnetic induction or magnetic field) $\vec{B}$

"The total number of magnetic field lines passing normally through a unit area is called magnetic flux density."

Mathematically, if ϕ_m is the magnetic flux linked with a surface of area S, then magnetic flux density is

$$B = \frac{\phi_m}{S}.$$ Its S.I unit is tesla (T) or wb/m^2

It is a vector quantity.

1. **Faraday's laws of electromagnetic induction:-**

On the basis of his experiment, Faraday gave the following two laws:-

(1) **Ist law:-** "Whenever the magnetic flux linked with a closed circuit or coil changes, an induced e.m.f is set up in the circuit. The induced e.m.f lasts as long as the change in magnetic flux continues."

(2) **2nd law:-** The magnitude of induced e.m.f is directly proportional to the rate of change of magnetic flux linked with the circuit.

If $d\phi_m$ is the change in magnetic flux in time dt second, then induced e.m.f (e) is given by,

$$e \propto \frac{d\phi_m}{dt} \quad or \quad e = -K\frac{d\phi_m}{dt} \quad \text{--- (1)}$$

where K is proportionality constant. And the negative sign shows that the induced e.m.f opposes the change in magnetic flux.

In S.I units, K = 1

$$\therefore \quad e = -\frac{d\phi_m}{dt}$$

For a coil of N-turns,

$$e = -N\frac{d\phi_m}{dt}$$

But $\phi_m = \iint_S \vec{B}\cdot\vec{ds}$

$$\therefore \quad e = -N\frac{d}{dt}\iint \vec{B}\cdot\vec{ds}$$

This is the mathematically form of Faraday's laws of e.m. induction.

2. **Integral form of Faraday's laws:-** Consider a surface S with C as its boundary as shown in fig (1). Let the magnetic field $\vec{B}$ is changing through the surface S with time. Then, the magnetic flux through the surface S at any time t is given by

$$\phi_m = \iint_S \vec{B}\cdot\vec{ds} \quad \text{--- (1)}$$

If e = induced e.m.f set up in the circuit, then according to Faraday's laws of e.m. induction

$$e = -\frac{d\phi_m}{dt} \quad \text{--- (2)}$$

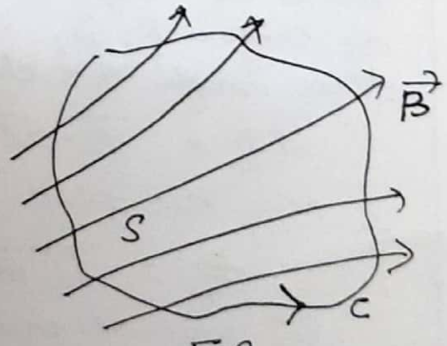


Fig (1)

using (1) in (2), we get

$$e = -\frac{d}{dt}\iint_S \vec{B}\cdot\vec{ds}$$

$$= -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot\vec{ds}$$

[$\because \vec{B}$ is a function of space and time - co-ordinates]

This is the integral form of Faraday's laws.

— x — x — x —

Differential form of Faraday's law of e.m. induction

[or Law of e.m. induction in universal form]

Consider a loop of wire placed in a non uniform magnetic field $\vec{B}$, as shown in fig ①

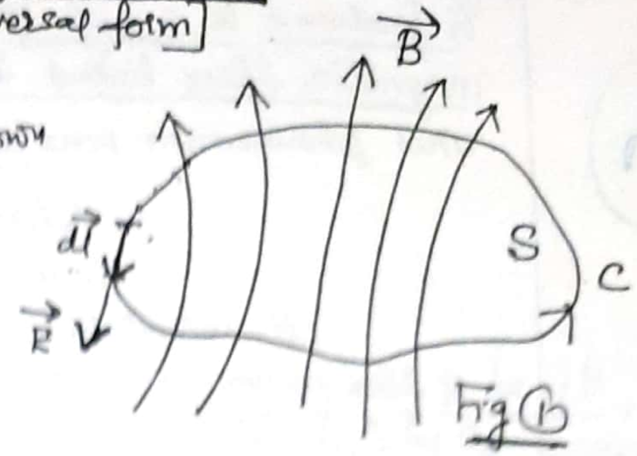


Fig ①

The total magnetic flux linked with the loop is given by

$$\phi_m = \iint_S \vec{B}\cdot\vec{ds} \quad \text{—①}$$

Differentiating ① w.r.t time, we get

$$\frac{d\phi_m}{dt} = \frac{d}{dt}\iint_S \vec{B}\cdot\vec{ds}$$

Since $\vec{B}$ is a function of both space and time co-ordinates

$$\therefore \frac{d\phi_m}{dt} = \iint_S \frac{\partial\vec{B}}{\partial t}\cdot\vec{ds} \quad \text{—②}$$

If e = induced e.m.f set up in the loop, then according to Faraday's laws of e.m. induction,

$$e = -\frac{d\phi_m}{dt}$$

using eqn ②, we get

$$e = -\iint_S \frac{\partial\vec{B}}{\partial t}\cdot\vec{ds} \quad \text{—③}$$

The e.m.f in the circuit is defined as the amount of work done in taking a unit +ive charge around the circuit and is given by line integral of electric field $\vec{E}$ around the circuit

i.e $e = \oint \vec{E}\cdot\vec{dl}$ —④

From ③ & ④, we get

$$\oint \vec{E}\cdot\vec{dl} = -\iint_S \frac{\partial\vec{B}}{\partial t}\cdot\vec{ds} \quad \text{—⑤}$$

According to Stoke's theorem,

$$\iint_S (\vec{\nabla}\times\vec{E})\cdot\vec{ds} = -\iint_S \frac{\partial\vec{B}}{\partial t}\cdot\vec{ds}$$

Since integrands on b/s are equal

$$\therefore \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}} \quad \text{—⑦}$$

This is the differential form of Faraday's laws of e.m. induction. It states that curl of $\vec{E}$ is equal to -ive rate of change of $\vec{B}$.]

This shows that the changing magnetic field produces an electric field. Thus, the source of induced electric field is varying magnetic field.

Eq ⑦ is also known as one of the Maxwell's equations of e.m field

Mutual inductance between two arbitrary circuits :- (OR Theorem of Reciprocity :-)

Consider two neighbouring circuits C_1 and C_2 as shown in fig ① Suppose the current I_1 flows through C_1. Let P be any point in the circuit – C_2. Then according according to Biot Savart's law, the magnetic field $d\vec{B}_1$ produced by the current element $I_1 d\vec{l}_1$ of the circuit C_1 at P is given by,

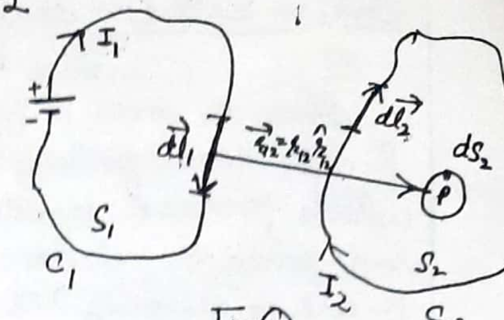


Fig ①

$$d\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \frac{d\vec{l}_1 \times \hat{r}_{12}}{r_{12}^2} \quad \text{—— (1)}$$

where r_{12} distance of observation point P from $d\vec{l}_1$

And $\hat{r}_{12}$ = unit vector pointing from $d\vec{l}_1$ to point P

∴ Total magnetic field $\vec{B}_1$ produced by the circuit – C_1 at P is

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint_{C_1} \frac{d\vec{l}_1 \times \hat{r}_{12}}{r_{12}^2} \quad \text{—— (2)}$$

The magnetic flux ϕ_2 linked with the circuit – C_2 is given by

$$\phi_2 = \iint_{S_2} \vec{B}_1 \cdot d\vec{S}_2 \quad \text{—— (3)}$$

where S_2 = area of the surface enclosed by C_2

Substituting ② in ③, we get

$$\phi_2 = \frac{\mu_0}{4\pi} I_1 \iint_{S_2} \oint_{C_1} \left(\frac{d\vec{l}_1 \times \hat{r}_{12}}{r_{12}^2} \right) \cdot d\vec{S}_2$$

But $\frac{\hat{r}_{12}}{r_{12}^2} = -\vec{\nabla}\left(\frac{1}{r_{12}}\right)$

$$\therefore \phi_2 = -\frac{\mu_0}{4\pi} I_1 \iint_{S_2} \oint_{C_1} \left[d\vec{l}_1 \times \vec{\nabla}\left(\frac{1}{r_{12}}\right) \right] \cdot d\vec{S}_2$$

$$\text{or } \phi_2 = \frac{\mu_0}{4\pi} I_1 \iint_{S_2} \oint_{C_1} \left[\vec{\nabla}\left(\frac{1}{r_{12}}\right) \times d\vec{l}_1 \right] \cdot d\vec{S}_2 \quad \text{—— (4)}$$

*

* using vector identity,

$$\vec{\nabla} \times \phi \vec{A} = \phi (\vec{\nabla} \times \vec{A}) + \vec{\nabla}\phi \times \vec{A}, \text{ we get}$$

$$\vec{\nabla} \times \frac{d\vec{l}_1}{r_{12}} = \frac{1}{r_{12}} (\vec{\nabla} \times d\vec{l}_1) + \vec{\nabla}\left(\frac{1}{r_{12}}\right) \times d\vec{l}_1$$

But $\vec{\nabla} \times d\vec{l}_1 = 0$

$$\therefore \vec{\nabla} \times \frac{d\vec{l}_1}{r_{12}} = \vec{\nabla}\left(\frac{1}{r_{12}}\right) \times d\vec{l}_1$$

Substituting this in eqn ④, we get

$$\phi_2 = \frac{\mu_0}{4\pi} I_1 \iint_{S_2} \oint_{C_1} \left[\vec{\nabla} \times \frac{d\vec{l}_1}{r_{12}} \right] \cdot d\vec{S}_2$$

$$\text{or } \phi_2 = \frac{\mu_0}{4\pi} I_1 \oint_{C_1} \iint_{S_2} \left(\vec{\nabla} \times \frac{d\vec{l}_1}{r_{12}} \right) \cdot d\vec{S}_2$$

According to Stoke's Theorem,

$$\iint_{S_2} \left(\vec{\nabla} \times \frac{d\vec{l}_1}{r_{12}} \right) \cdot d\vec{S}_2 = \oint_{C_2} \frac{d\vec{l}_1}{r_{12}} \cdot d\vec{l}_2$$

where dl_2 = elementary length of circuit – C_2.

$$\therefore \phi_2 = \frac{\mu_0}{4\pi} I_1 \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r_{12}} \quad \text{—— (5)}$$

Comparing eqn ⑤ with

$\phi_2 = M_{21} I_1$, we get

$$M_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r_{12}} \quad \text{—— (6)}$$

This gives the mutual inductance of circuit C_2 due to current I_1 in circuit – C_1

— contd —

Similarly, mutual inductance (M_{12}) of circuit c_1 due to current I_2 in circuit c_2 is given by,

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{c_2} \oint_{c_1} \frac{\vec{dl_2} \cdot \vec{dl_1}}{r_{21}} \quad (7)$$

Since $r_{12} = r_{21}$ and $\vec{dl_1} \cdot \vec{dl_2} = \vec{dl_2} \cdot \vec{dl_1}$

$\therefore$ From eqs (6) and (7), we get

$$M_{12} = M_{21} = M \text{ (say)}$$

This relation is known as Theorem of Reciprocity. It states that the mutual inductance is symmetric.

From eqns (6) and (7) it is found that mutual inductance depends on the geometry of the two coils.

Relation between Self and mutual inductances :-

Consider two coils c_1 and c_2 placed close to each other.

Let :- L_1, L_2 = Self inductances of the two coils resp.

I_1, I_2 = currents in the two coils resp.

ϕ_1 = magnetic flux linked with coil c_1 due to current I_1 in it

ϕ_2 = magnetic flux lined with coil c_2 due to current I_2 in it

Then $\phi_1 = L_1 I_1$ — (1)

and $\phi_2 = L_2 I_2$ — (2)

Let M_{21} = coefficient of mutual induction of coil c_2 w.r.t coil c_1

Then, the magnetic flux linked with c_2 due to mutual induction when current I_1 is passed through c_1 is

$\phi_{21} = M_{21} I_1$ — (3)

But the coupling of the coils is so adjusted that whole of the flux produced in c_1 is linked with c_2

$\therefore \phi_{21} = \phi_1$

or $M_{21} I_1 = L_1 I_1$

or $\boxed{M_{21} = L_1}$ — (4)

Similarly, the flux due to self induction of c_2 is equal to the flux linked with c_1 due to mutual induction of c_2 on c_1

i.e. $\phi_{12} = \phi_2$

or $M_{12} I_2 = L_2 I_2$

or $\boxed{M_{12} = L_2}$ — (5)

Multiplying (4) and (5), we get

$M_{21} M_{12} = L_1 L_2$

But from Reciprocity theorem,

$M_{21} = M_{12} = M$

$\therefore M^2 = L_1 L_2$

or $\boxed{M = \sqrt{L_1 L_2}}$

This is the relation between mutual and self inductances of two coils.

Coefficient of Coupling (K)

When whole of the magnetic flux produced due to current in one coil is linked with the other coil, then the relation between mutual induction and self inductances of two coils is given by,

$$M = \sqrt{L_1 L_2} \quad \text{--- (1)}$$

Where L_1, L_2 = Self inductances of two coils

M = Mutual inductance of two coils.

But in actual practice, there is some leakage of flux and only a fraction of flux associated with one coil is linked with the other coil. Under such conditions,

$$M = k\sqrt{L_1 L_2} \quad \text{--- (2)}$$

where k is a number and is called coefficient of coupling.

Its value depends on the geometry and relative positions of the coils.

The value of k lies between 0 and 1.

The value of k = 0 means no coupling and k=1 means perfect coupling. Perfect coupling means whole of the flux produced by one coil is linked with the other coil. That is there is no leakage of flux. And no coupling means that flux produced by one coil is not at all linked with the other coil.

displacement vector

Energy in electromagnetic fields [or Pointing Theorem]

Consider an electromagnetic wave travelling along x-axis with electric and magnetic field vectors $\vec{E}$ and $\vec{B}$ along y and z axes respectively. Thus the wave propagates in the direction of pointing vector $\vec{S}$, which is given by

$$\vec{S} = \vec{E} \times \vec{H} \quad \text{——} (1)$$

Take the divergence of pointing vector $\vec{S}$ in free space, we get

$$\vec{\nabla} \cdot \vec{S} = \vec{\nabla} \cdot (\vec{E} \times \vec{H})$$

$$= \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H})$$

$\because \vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

Since $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

And $\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

$\because \vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t})$

or $\vec{\nabla} \times \mu_0 \vec{H} = \mu_0 (\vec{J} + \partial \vec{D}/\partial t)$

or $\vec{\nabla} \times \vec{H} = \vec{J} + \partial \vec{D}/\partial t$

Since in free space $\vec{J} = 0$

$\therefore \vec{\nabla} \times \vec{H} = \partial \vec{D}/\partial t$

$$\therefore \vec{\nabla} \cdot \vec{S} = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

$$= -\left[\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}\right]$$

$$= -\left[\epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{H} \cdot \frac{\partial \vec{H}}{\partial t}\right]$$

$\because \vec{D} = \epsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$

$$= -\left[\frac{1}{2} \epsilon_0 \, 2\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{1}{2} \mu_0 \, 2\vec{H} \cdot \frac{\partial \vec{H}}{\partial t}\right]$$

$$= -\left[\frac{1}{2} \epsilon_0 \frac{\partial}{\partial t}(\vec{E} \cdot \vec{E}) + \frac{1}{2} \mu_0 \frac{\partial}{\partial t}(\vec{H} \cdot \vec{H})\right]$$

$\because \frac{\partial}{\partial t}(\vec{E} \cdot \vec{E}) = \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = 2\vec{E} \cdot \partial \vec{E}/\partial t$

$$= -\left[\frac{1}{2} \epsilon_0 \frac{\partial (E^2)}{\partial t} + \frac{1}{2} \mu_0 \frac{\partial}{\partial t}(H^2)\right]$$

$$= -\frac{\partial}{\partial t}\left[\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2\right] \quad \text{——} (2)$$

Now consider a volume V enclosed by a surface S. Integrating eq (2) over volume V, we get

$$\iiint_V (\vec{\nabla} \cdot \vec{S}) \, dV = -\frac{\partial}{\partial t} \iiint_V \left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2\right) dV$$

— Contd —

pplying Gauss's divergence theorem to L.H.S

of above eq., we get

$$\oiint_S \vec{S}\cdot d\vec{S} = -\frac{\partial}{\partial t}\iiint_V \left[\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right] dV \quad \text{—— (3)}$$

Since $\frac{1}{2}\epsilon_0 E^2$ = electrostatic energy density (u_e)

and $\frac{1}{2}\mu_0 H^2$ = magnetostatic energy density (u_m)

∴ The term within the integral on R.H.S of eqn (3) represents the sum of energy densities of electric and magnetic. This sum is called energy density of e.m. field (u_{em})

Thus, $\boxed{u_{em} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2}$ —— (4)

∴ Total energy U_{em} of e.m. field over volume V enclosed by surface S is

$$U_{em} = \iiint_V \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right) dV \quad \text{— (5)}$$

Thus the R.H.S of eqn (3) represents the rate of flow of energy of e.m. field over the volume V.

∴ $\oiint_S \vec{S}\cdot d\vec{S}$ = Rate of flow of energy across the boundary of volume V.

Eq. (3) shows that power flux through a closed area is equal to the rate of outflow of energy from the volume enclosed by the area. This is Pointing theorem in free space. This is also known as Law of conservation of energy in case of e.m. wave.

Pointing vector ($\vec{S}$) [or Flux vector] :-

"It is defined as the cross product of electric field vector $\vec{E}$ and magnetic field vector $\vec{H}$."

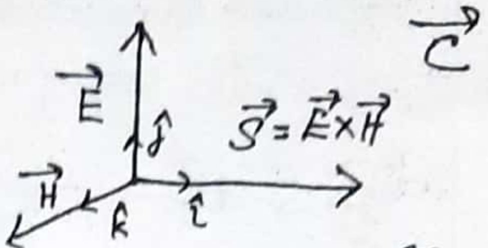


Fig ①

Mathematically,

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \quad [\because \vec{B} = \mu_0 \vec{H}]$$

Suppose an electromagnetic wave is propagating along x-axis with $\vec{E}$ along y-axis and $\vec{H}$ along z-axis as shown in fig ①

Then,

$$\vec{S} = \hat{j} E_y \times \hat{k} H_z = (\hat{j} \times \hat{k}) E_y H_z$$

$$= \hat{i} E_y H_z$$

But $E_y H_z = \frac{\text{Voltage}}{\text{length}} \times \frac{\text{current}}{\text{length}} = \frac{VI}{\text{Area}} = \frac{\text{Electric power}}{\text{Area}}$

or $E_y H_z = \frac{\text{Electric power} \times \text{time}}{\text{time} \times \text{area}} = \frac{\text{Electric energy}}{\text{time} \times \text{area}}$

$\therefore \vec{S} = \hat{i}\left(\frac{\text{Electric power}}{\text{Area}}\right) = \hat{i}\left(\frac{\text{Electric energy}}{\text{time} \times \text{area}}\right)$

Thus pointing vector $\vec{S}$ measures the flow of electric energy per unit time per unit area (of the medium) held perpendicular to the direction of propagation of the wave. It is also called flux vector.

Unit :- The S.I unit of $\vec{S}$ is $J s^{-1} m^{-2}$ or Watt/m^2.